Finding the Determinant of Complement of Trees with Diameter less than Five

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Abstract. A graph G is said to be singular if its adjacency matrix is singular; otherwise it is said to be non-singular. In this paper, we using the results of determinant on looped-trees to compute the determinant of the complement of a certain class of trees with diameters 3 and 4.

1. Introduction and Preliminaries

A graph is completely determined by its adjacencies. This information can be conveniently stated in matrix form. Indeed, with a given graph, adequately labeled, there are associated several matrices, including the adjacency matrix, cycle matrix, and cocycle matrix. It is often possible to make use of these matrices in order to identify certain properties of a graph. The classic theorem on graphs and matrices is the Matrix-Tree Theorem, which gives the number of spanning trees in any labeled graph (see [1]).

The results of adjacency matrix with non-singularity of tree and tree complement are following. In 1989, S. V. Gervacio and H. M. Rara [2] investigated non-singularity of trees. In 1996, S. V. Gervacio [3] was determined the singularity or non-singularity of the complement of a tree with diameter less than 5, indeed, for a graph tree T, $\overline{T}$ is singular if diameter of tree is 0, 1, or 2 i.e. $\det A(\overline{T}) = 0$. Recently. In 2015, N. Pipattanajinda and Y. Kim [4, 5] obtained the determinant of the complement of a tree with diameter 5, and determined the singularity or non-singularity of the complement of a certain class of trees with diameter 5. Furthermore, the determinant of the complement of a tree with diameter 4 is not found. In this paper, we will use the results of [5], determinant on looped-trees, to compute the determinant of complement of trees with diameters 3 and 4.

By a graph G, we mean a pair (V(G), E(G)), where V(G) is a finite non-empty set of elements called vertices and $E(G) \subseteq \{\{u, v\} | u, v \in V(G)\}$, the set of 2-subsets of V(G) whose elements are called edges. The edge $\{u, u\}$ in a graph G is called loop. A graph G is a simple graph if it has no loops and no more than one edge between any two different vertices. For a simple graph G = (V(G), E(G)), a graph $G^o = (V(G^o), E(G^o))$ with $V(G^o) = V(G)$ and $E(G^o) = \{\{u, v\} | \{u, v\} \in E(G)\} \cup \{\{u, u\} | u \in V(G^o)\}$ is called a looped-graph of G. In particular, if G is a tree, G^o is called a looped-tree.

If G is a graph with vertices $x_1, x_2, \ldots, x_n$, we define the adjacency matrix of G to be the $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = \begin{cases} 1 & \text{if } \{x_i, x_j\} \in E(G) \\ 0 & \text{otherwise.} \end{cases}$ The graph G is said to be singular if A(G) is singular, i.e., det A(G) = 0; otherwise G is said to be non-singular. If $S \subset V(G)$, then $G \setminus S$ denotes the graph obtained from G by deleting all the vertices $x \in S$. The complement $\overline{G}$ of G is a graph such that $V(\overline{G}) = V(G)$ and $\{u, v\} \in E(\overline{G})$ if and only if $\{u, v\} \notin E(G)$ for any $u, v \in V(G)$ and $u \neq v$. The loop complement $\overline{G}^o$ of G is a graph such that $V(\overline{G}^o) = V(G)$ and $\{u, v\} \in E(\overline{G}^o)$ if and only if $\{u, v\} \notin E(G)$ for any $u, v \in V(G)$. When G is a simple graph, the loop complement of G is the complement of G^o, that is, $\overline{G}^o = \overline{G^o}$, and the loop complement of G^o is the complement of G, that is, $\overline{G^o}^o = G$. Other terms whose definitions are not given here may be found in many graph theory books, e.g., [1].

Following from [5], for positive integers $m, r, s, m_1, m_2, \cdots, m_r$, we define a series of looped-trees, $T^0_{2:1}$, $T^0_{2:m}$, $T^0_{3:r,s}$ and $T^0_{4:m_1,\ldots,m_r}$ as follows: by $T^0_{2:1}$ we mean a looped-tree of a tree with diameter $\le$ 2, which is depicted in fig. 1 where x is called the central vertex, and m is the number of vertices but the central vertex w. For two disjoint looped-trees $T^0_{2:r}$, $T^0_{2:s}$ as follows*:* by with central vertices x, y respectively, we form a looped-tree $T^0_{3:r,s}$ by joining two central vertices as shown in fig. 1, where x, y is called central vertices of $T^0_{3:r,s}$. For disjoint looped-trees $T^0_{2:m_1}, T^0_{2:m_2}, \ldots, T^0_{2:m_r}$, with central vertices $y_1, y_2, \ldots, y_r$ respectively, we form a looped-tree $T^0_{4:m_1,\ldots,m_r}$ by joining all central vertices y_i to a new vertex x (see fig. 2 where x is called the central vertex of $T^0_{4:m_1,\ldots,m_r}$).

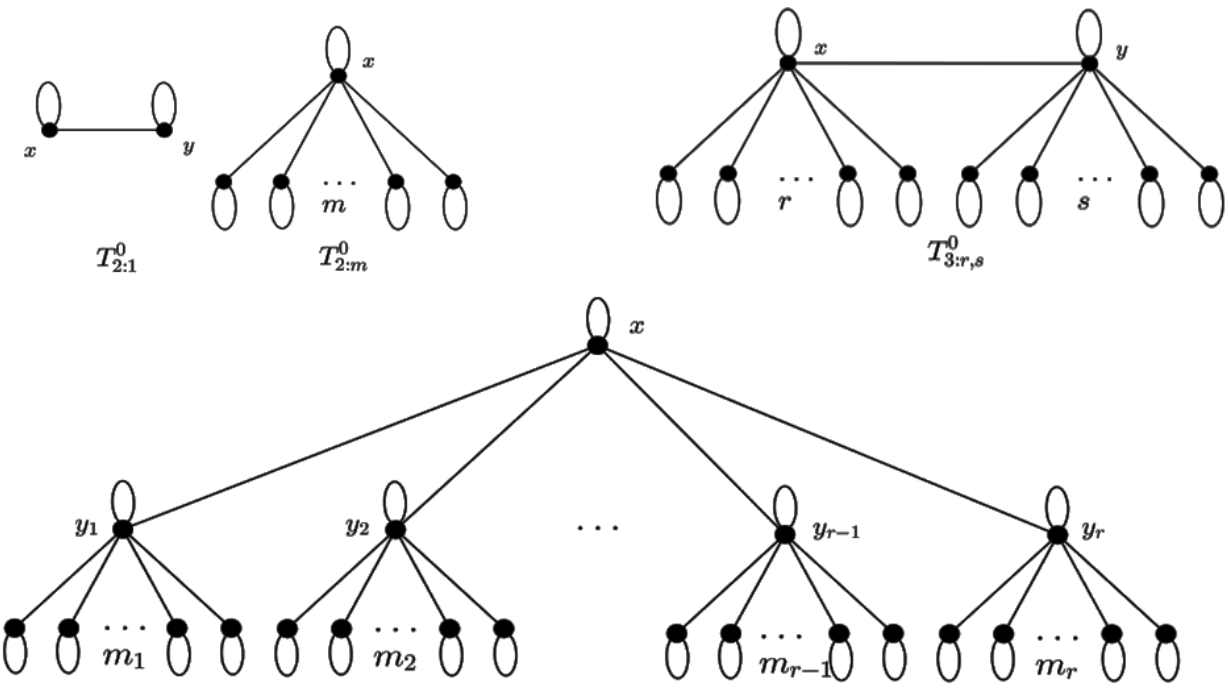

Fig. 1. Graph $T^0_{2:m}, T^0_{3:r,s}$ and $T^0_{4:m_1,\ldots,m_r}$

Next, the following results have been proved.

Lemma 1.1 [5] The determinant of looped-trees $T^0_{2:m}, T^0_{3:r,s}$ and $T^0_{4:m_1,\ldots,m_r}$ are following.

(i) $|A(T^0_{2:m})| = 1 - m$,

(ii) $|A(T^0_{3:r,s})| = rs - r - s$, and

(iii) $|A(T^0_{4:m_1,\ldots,m_r})| = \prod_{i=1}^{r}(1 - m_i) - \sum_{i=1}^{r} \frac{(1-m_1)(1-m_2)\cdots(1-m_r)}{(1-m_i)}$.

For the complete graph $K_k^{(1)}$ of order k $(k \ge 1)$ with 1 loop, and a graph G of order n, the following property was shown in [6], where $K_k^{(1)} + \overline{G}^o$ means the join of $K_k^{(1)}$ and $\overline{G}^o$.

Lemma 1.3 [6] Let G be a graph of order n. Then $|A(G)| = (-1)^{n+k-1}|A(K_k^{(1)} + \overline{G}^o)|$.

Let G be a graph whose vertices are $v_1, v_2, \ldots$ and let every edge be associated with the variable w_i. Then we can construct a variable adjacency matrix A(G,w) for the graph G as follows: the (i, j) entry is w_k if and only if $\{v_i, v_j\} \in E(G)$ and the variable w_k is associated with edge $\{v_i, v_j\}$, and this entry is 0 if $\{v_i, v_j\} \notin E(G)$. We note that the ordinary adjacency matrix A(G) is obtained from A(G,w) by substituting $w_k = 1$ for each of the variables for the edges of G. Let G be a graph. An (ordinary) linear subgraph of G is a spanning subgraph whose components are lines or cycles. Further, let n be the number of linear subgraphs of G and let G_i be the ith linear subgraph. In [3], F. Harary showed the following theorem. We note that a simple observation gives of the theorem works for our case in which the components of a linear subgraph contain loops.

Lemma 1.4 [7] Let G be a graph. Then

$$|A(G,w)| = \sum_{i=1}^{n} |A(G_i,w)| \text{ and } |A(G,w)| = \sum_{i=1}^{n} (-1)^{e_i} 2^{c_i} \prod_{w_k \in L_i} w_k^2 \prod_{w_j \in M_i} w_j \ ,$$

where (1) e_i is the number of even components of G_i, (2) c_i is the number of components of G_i containing more than two points, and thus consisting of a single undirected cycle, (3) L_i is the set of components of G_i consisting of two points and the line joining them, and (4) M_i is the remaining components of G_i each of which is a cycle.

2. The Determinant of Complement of Trees with Diameter 3

For a looped-tree $T^0_{3:r,s}$ with central vertices x and y and $z \notin V(T^0_{3:r,s})$, denoted $T^0_{3:r,s} \ {}^{x}_{y}\!\!> z^0$ by the graph with $V(T^0_{3:r,s} \ {}^{x}_{y}\!\!> z^0) = V(T^0_{3:r,s}) \cup \{z\}$ and $E(T^0_{3:r,s} \ {}^{x}_{y}\!\!> z^0) = E(T^0_{3:r,s}) \cup \{\{x,z\},\{y,z\},\{z,z\}\}$.

Lemma 2.1 For positive integers r and s,

$$\left|A(\overline{T_{3:r,s}})\right| = (-1)^t \left|A(T^0_{3:r,s} \ {}^{x}_{y}\!\!> z^0, w)\right|$$

where the values associated with a loop at z, the edge $\{x,z\}$ and the edge $\{y,z\}$ are $1-(r+s)$, $1-r$ and $1-s$ respectively, and every other edge has the value 1, and $t = r+s+2$ is the order of $T^0_{3:r,s}$.

Proof. From Lemma 1.3, $\left|A(\overline{T_{3:r,s}})\right| = (-1)^t \left|A(T^0_{3:r,s} + z^0)\right|$, where $t = r+s+2$ is the order of $T^0_{3:r,s}$. We note that the adjacency matrix of $T^0_{3:r,s} + z^0$ follow this:

$$A(T^0_{3:r,s} + z^0) = \begin{array}{c} \\ x \\ y \\ x_1 \\ \vdots \\ x_r \\ y_1 \\ \vdots \\ y_s \\ z^0 \end{array} \begin{array}{c} \begin{array}{cccccccccc} x & y & x_1 & \cdots & x_r & y_1 & \cdots & y_s & z^0 \end{array} \\ \left(\begin{array}{cc|ccc|ccc|c} 1 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 \\ \hline 1 & 0 & & & & & & & 1 \\ \vdots & \vdots & & I_r & & & 0 & & \vdots \\ 1 & 0 & & & & & & & 1 \\ \hline 0 & 1 & & & & & & & 1 \\ \vdots & \vdots & & 0 & & & I_s & & \vdots \\ 0 & 1 & & & & & & & 1 \\ \hline 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \end{array}\right) \end{array}$$

By subtract rows was corresponding to $\mathbf{x_1},\ldots,\mathbf{x_r}$ from the last row corresponding to z. Similarly, by subtract rows was corresponding to $\mathbf{y_1},\ldots,\mathbf{y_s}$ from the last row corresponding to z. Then we now subtract columns corresponding to $\mathbf{x_1},\ldots,\mathbf{x_r},\mathbf{y_1},\ldots,\mathbf{y_s}$ from the last column to get:

$$|A(T^0_{3:r,s} + z^0)| = \det\left(\begin{array}{cc|ccc|ccc|c} 1 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 1-r \\ 1 & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1-s \\ \hline 1 & 0 & & & & & & & 0 \\ \vdots & \vdots & & I_r & & & 0 & & \vdots \\ 1 & 0 & & & & & & & 0 \\ \hline 0 & 1 & & & & & & & 0 \\ \vdots & \vdots & & 0 & & & I_s & & \vdots \\ 0 & 1 & & & & & & & 0 \\ \hline 1-r & 1-s & 0 & \cdots & 0 & 0 & \cdots & 0 & 1-(r+s) \end{array}\right)$$

$$= \left|A(T^0_{3:r,s} \ {}^{x}_{y}\!\!> z^0, w)\right|$$

where the values associated with a loop at z, the edge $\{x,z\}$ and the edge $\{y,z\}$ are $1-(r+s)$, $1-r$ and $1-s$, respectively, and every other edge has the value 1.

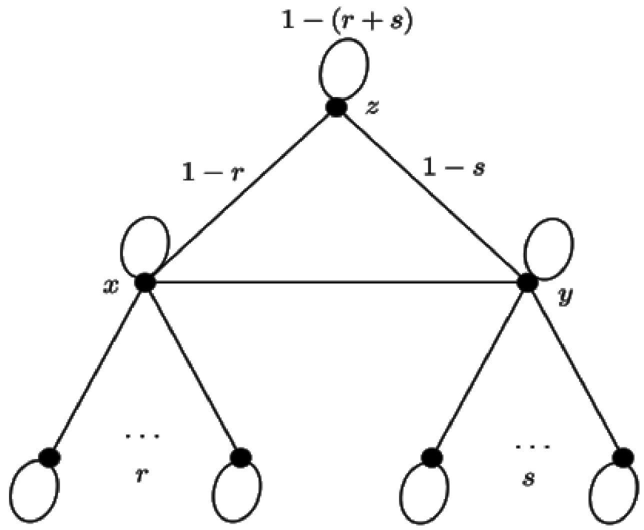

Fig. 2. Graph $(T^0_{3:r,s}\ {}^{x}_{y}\!\!> z^0, w)$ of Lemma 2.1

Theorem 2.2 For positive integers r and s,

$$\left|A(\overline{T_{3:r,s}})\right| = (-1)^t rs$$

where t is the order of $T^0_{3:r,s}$.

Proof. By applying Lemma 2.1, we have $\left|A(\overline{T_{3:r,s}})\right| = (-1)^t\left|A(T^0_{3:r,s}\ {}^{x}_{y}\!\!> z^0, w)\right|$, where t is the order of $T^0_{3:r,s}$.

We partition the set of all linear subgraphs of $T^0_{3:r,s}\ {}^{x}_{y}\!\!> z^0$ into 4 classes $\mathscr{H}_1, \mathscr{H}_2, \mathscr{H}_3$ which consists of all linear subgraphs containing a cycle {x, y, z}, a line {x, z}, and a line {y, z}, respectively, and $\mathscr{H}_4$ consist of all linear subgraphs containing neither {x, z} or {y, z} nor a cycle {x, y, z}**.** By Theorem 1**.**4, we have $\left|A(\overline{T_{3:r,s}})\right| = (-1)^t\left|A(T^0_{3:r,s}\ {}^{x}_{y}\!\!> z^0, w)\right| = (-1)^t\sum_{i=1}^{4}\left(\sum_{H\in\mathscr{H}_i}|A(H,w)|\right)$.

(1) Consider $\sum_{H\in\mathscr{H}_1}|A(H,w)|$. From $\mathscr{H}_1$ is the class which consists of all linear subgraphs containing a cycle {x, y, z}, then $\sum_{H\in\mathscr{H}_1}|A(H,w)| = \begin{vmatrix} 0 & 1 & 1-r \\ 1 & 0 & 1-s \\ 1-r & 1-s & 0 \end{vmatrix} = 2(1-r)(1-s)$, see fig. 3(a).

(2) Consider $\sum_{H\in\mathscr{H}_2}|A(H,w)|$. From $\mathscr{H}_2$ is the class which consists of all linear subgraphs containing a line {x, z}, then $\sum_{H\in\mathscr{H}_2}|A(H,w)| = \begin{vmatrix} 0 & 1-r \\ 1-r & 0 \end{vmatrix}|T^0_{2:s}| = -(1-r)^2|T^0_{2:s}|$, see fig. 3(b).

(3) Consider $\sum_{H\in\mathscr{H}_3}|A(H,w)|$. From $\mathscr{H}_3$ is the class which consists of all linear subgraphs containing a line {y, z}, then $\sum_{H\in\mathscr{H}_3}|A(H,w)| = \begin{vmatrix} 0 & 1-s \\ 1-s & 0 \end{vmatrix}|T^0_{2:s}| = -(1-s)^2|T^0_{2:r}|$, see fig. 3(c).

(4) Consider $\sum_{H\in\mathscr{H}_3}|A(H,w)|$. From $\mathscr{H}_3$ is the class which consisting of all linear subgraphs containing neither {x, z} or {y, z} nor a cycle {x, y, z}, then $\sum_{H\in\mathscr{H}_2}|A(H,w)| = (1-(r+s))\left|T^0_{3:r,s}\right|$, see fig. 3(d).

From (1) – (4) and Lemma 1.1 (i) and (ii), $\left|A(\overline{T_{3:r,s}})\right|$

$$= (-1)^t\left(2(1-r)(1-s) - (1-r)^2|T^0_{2:s}| - (1-s)^2|T^0_{2:r}| + \left(1-(r+s)\right)\left|T^0_{3:r,s}\right|\right)$$

$$= (-1)^t\Big(2(1-r)(1-s) - (1-r)^2(1-s) - (1-s)^2(1-r) + \left(1-(r+s)\right)(rs-r-s)\Big)$$

$$= (-1)^t\Big((1-r)(1-s)(r+s) + \big(1-(r+s)\big)(rs-r-s)\Big)$$

$$= (-1)^t rs.$$

3. The Determinant of Complement of Trees with Diameter 4

For a looped-tree $T^0_{4:m_1,\dots,m_r}$ with central vertex x and $z \notin V(T^0_{4:m_1,\dots,m_r})$, denoted $T^0_{4:m_1,\dots,m_r}x{\sim}z^0$ by the graph with $V\big(T^0_{4:m_1,\dots,m_r}x{\sim}z^0\big) = V\big(T^0_{4:m_1,\dots,m_r}\big) \cup \{z\}$ and $E\big(T^0_{4:m_1,\dots,m_r}x{\sim}z^0\big) = E\big(T^0_{4:m_1,\dots,m_r}\big) \cup \{\{x,z\},\{z,z\}\}$.

Lemma 3.1 For positive integers $m_1, m_2, \dots, m_r$,

$$\left|A(\overline{T_{4:m_1,\dots,m_r}})\right| = (-1)^t\left|A(T^0_{4:m_1,\dots,m_r}x{\sim}z^0, w)\right|$$

where the values associated with both a loop at z and the edge $\{x,z\}$ is $1-r$, and every other edge has the value 1, and t is the order of $T^0_{4:m_1,\dots,m_r}$.

Proof. From Lemma 1.3, $\left|A(\overline{T_{4:m_1,\dots,m_r}})\right| = (-1)^t\left|A(T^0_{4:m_1,\dots,m_r} + z^0)\right|$, where t is the order of $T^0_{4:m_1,\dots,m_r}$. We note that the adjacency matrix of $T^0_{4:m_1,\dots,m_r} + z^0$ is of the following form:

$$A\big(T^0_{4:m_1,\dots,m_r} + z^0\big) = \begin{array}{c} \\ x \\ y_1 \\ M_1 \\ y_2 \\ M_2 \\ \vdots \\ y_r \\ M_r \\ z^0 \end{array}\!\!\begin{array}{c} \begin{array}{ccccccccc} x & y_1 & M_1 & y_2 & M_2 & \dots & y_r & M_r & z^0 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & I_{m_1} & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & I_{m_2} & \cdots & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & I_{m_r} & 1 \\ 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{pmatrix} \end{array}$$

By subtracting rows corresponding to $\mathbf{y_1}, \dots, \mathbf{y_r}$ from the last row corresponding to z, and we now subtract columns corresponding to $\mathbf{y_1}, \dots, \mathbf{y_r}$ from the last column to get

$$\left|A\big(T^0_{4:m_1,\dots,m_r} + z^0\big)\right| = \det\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 & 1-r \\ 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & I_{m_1} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & I_{m_2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & I_{m_r} & 0 \\ 1-r & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1-r \end{pmatrix}$$

$$= \left|A(T^0_{4:m_1,\dots,m_r}x{\sim}z^0, w)\right|$$

where the values associated with both a loop at z and the edge $\{x,z\}$ is $1-r$, and every other edge has the value 1.

Theorem 3.2 For positive integers $m_1, m_2, \dots, m_r$,

$$\left|A(\overline{T_{4:m_1,\dots,m_r}})\right| = (-1)^t(1-r)\left[r\prod_{i=1}^{r}(1-m_i) - \sum_{i=1}^{r}\frac{(1-m_1)(1-m_2)\cdots(1-m_r)}{(1-m_i)}\right],$$

where t is the order of $T^0_{4:m_1,\dots,m_r}$.

Proof. By applying Lemma 3.1, we have $\left|A(\overline{T_{4:m_1,\ldots,m_r}})\right| = (-1)^t\left|A(T^0_{4:m_1,\ldots,m_r}x{\sim}z^0, w)\right|$, where t is the order of $T^0_{4:m_1,\ldots,m_r}$.

We partition the set of all linear subgraphs of $T^0_{4:m_1,\ldots,m_r}x{\sim}z^0$ into 2 classes $\mathscr{H}_1$ which consists of all linear subgraphs containing a line {x, z}, and $\mathscr{H}_2$ consisting of all linear subgraphs non-containing line {x, z}. By Theorem 1.4, we have:

$$\left|A\left(\overline{T_{4:m_1,\ldots,m_r}}\right)\right| = (-1)^t\left|A\left(T^0_{4:m_1,\ldots,m_r}x{\sim}z^0, w\right)\right| = (-1)^t\sum_{i=1}^{2}\left(\sum_{H\in\mathscr{H}_i}|A(H,w)|\right).$$

Consider $\sum_{H\in\mathscr{H}_1}|A(H,w)|$. From $\mathscr{H}_1$ is the class which consists of all linear subgraphs containing a line {x, z}, then $\sum_{H\in\mathscr{H}_1}|A(H,w)| = \begin{vmatrix} 0 & 1-r \\ 1-r & 0 \end{vmatrix}\left|T^0_{2:m_1}\right|\left|T^0_{2:m_2}\right|\cdots\left|T^0_{2:m_r}\right| = -(1-r)^2\left|T^0_{2:m_1}\right|\left|T^0_{2:m_2}\right|\cdots\left|T^0_{2:m_r}\right|$.

From Lemma 1.1 (i), $\sum_{H\in\mathscr{H}_1}|A(H,w)| = -(1-r)^2\prod_{i=1}^{r}(1-m_i)$.

Next, consider $\sum_{H\in\mathscr{H}_2}|A(H,w)|$. From $\mathscr{H}_2$ is the class which consists of all linear subgraphs non-containing line {x, z}, then $\sum_{H\in\mathscr{H}_2}|A(H,w)| = (1-r)\left|T^0_{4:m_1,\ldots,m_r}\right|$.

From Lemma 1.1 (iii), $\sum_{H\in\mathscr{H}_2}|A(H,w)| = (1-r)\left(\prod_{i=1}^{r}(1-m_i) - \sum_{i=1}^{r}\frac{(1-m_1)(1-m_2)\cdots(1-m_r)}{(1-m_i)}\right)$.

Therefore,

$$\begin{aligned}\left|A\left(\overline{T_{4:m_1,\ldots,m_r}}\right)\right| &= (-1)^t\left(-(1-r)^2\prod_{i=1}^{r}(1-m_i) + (1-r)\left(\prod_{i=1}^{r}(1-m_i) - \sum_{i=1}^{r}\frac{(1-m_1)(1-m_2)\cdots(1-m_r)}{(1-m_i)}\right)\right),\\ &= (-1)^t(1-r)\left(r\prod_{i=1}^{r}(1-m_i) - \sum_{i=1}^{r}\frac{(1-m_1)(1-m_2)\cdots(1-m_r)}{(1-m_i)}\right).\end{aligned}$$

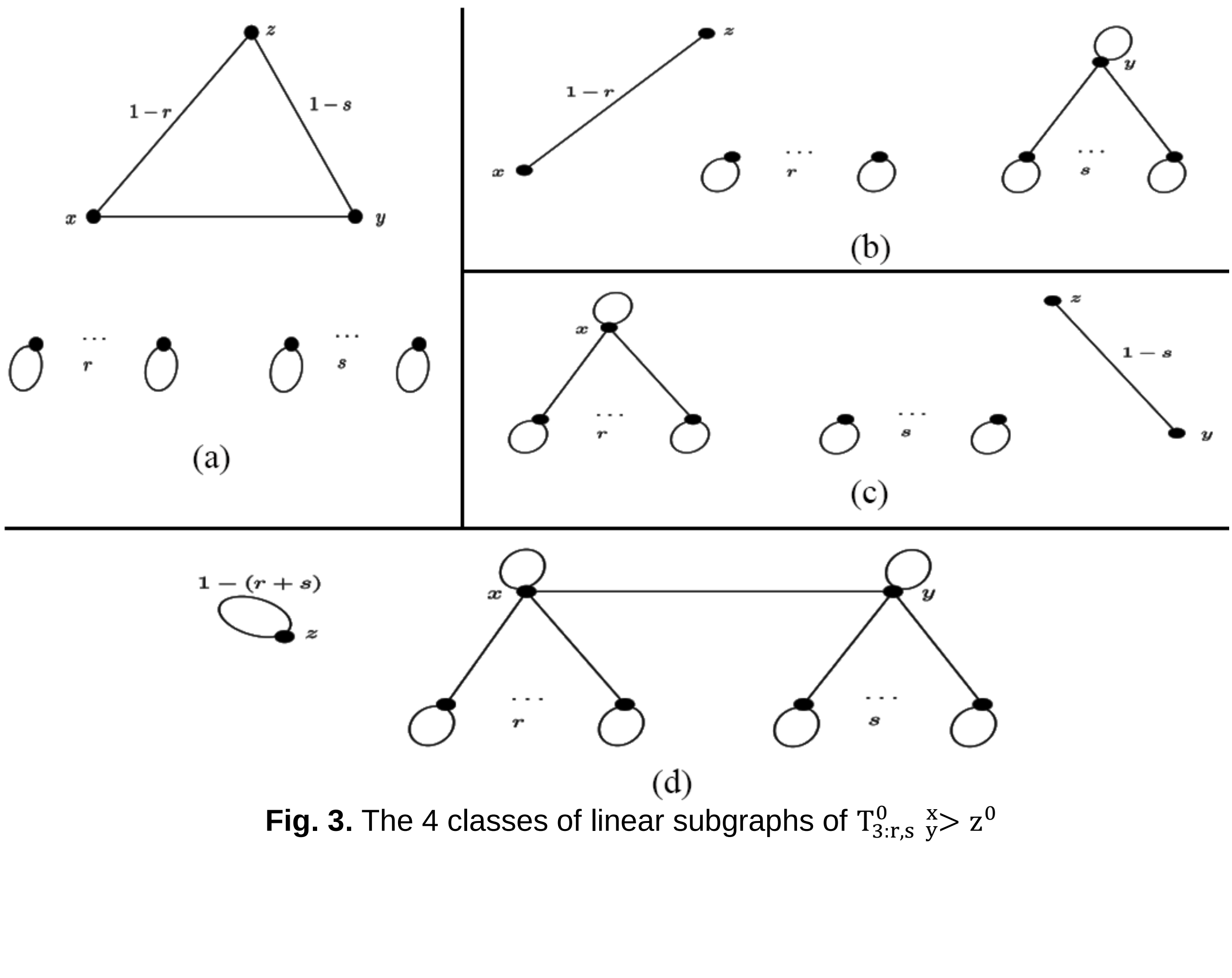

Fig. 3. The 4 classes of linear subgraphs of $T^0_{3:r,s}\ {}^{x}_{y}{>}\ z^0$

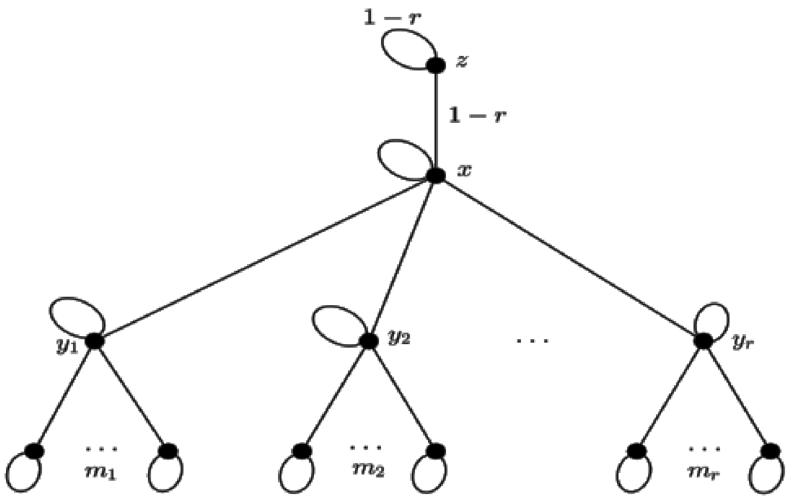

Fig. 4. Graph $(T^0_{4:m_1,\ldots,m_r}x\sim z^0, w)$ of Lemma 3.1

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