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# Correctness Proof of Min-Plus Algebra for Network Shortest Paths Simultaneous Calculation

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**Abstract.** The “network quality” is one of the mass transit network main quality, apart from the operational quality. The connectivity and the accessibility quality are two important aspects of a “network quality”. This analysis needs the whole point to point shortest path data. A special matrix calculation method for all shortest paths simultaneous calculation has been developed. A mathematical proof need to be developed. To be correct mathematically, the calculation method must satisfy a certain algorithm requirements. It has been proofed that the Min-Plus Algebra ensure the algorithm and is thus capable to always give the correct answer. The Min-Plus Algebra for Network Shortest Path Simultaneous Calculation is mathematically correct.

## 1. Introduction

Indonesian big cities must be considered as late to provide urban public mass transport. Having a good mass transport network planning and evaluation are therefore important. The first step to develop an urban mass transit is by developing a master plan, followed by a step by step line development and implementation. It can be seen easily that “network quality” assessment is capital. A “network quality” analysis need enormous network calculation. One of essential major data needed is the entire shortest path data [1-8].

Shortest Path calculation method has been created and developed largely. In Operation Research and Discrete Mathematics, it has been started by Dijkstra algorithm down to Floyd algorithm for simultaneous calculation, by passing, among others, through Kruskal-Wallis algorithm [9,10]. But, these all are still based on list of link’s data. The method can be found also in Graph Theory [11]. Practically, all of these are not based on matrix calculation and always based on list of link’s data. Its practicality still can be improved.

Spreadsheet is a very powerful and yet very versatile software to execute calculation. The spreadsheet is very excellent to be used for any kind of matrix calculation. Sophisticated softwares for transportation modelling can be found easily, but are not always easily accesible for financial reason. Operation research softwares are the same.

*So, why do not we develop a special network calculation method designated to be easily used on a spreadsheet software and also to be easily written in programming language ?*. Fortunately, a Special Matrix Technique for Transportation Network Analyse (SMT for TNA) has been developed. This is specially intended, designed and developed to be easily used on spreadsheet software [2-7]. The classic Matrix Theory can not be used for this purpose [12].

One of its major matrix calculation technique is the Min-Plus Algebra for All Shortest Paths Simultaneous Calculation [5-7]. The Min-Plus Algebra is a development of the Max-Plus Algebra. The Max-Plus Algebra was first developed by INRIA in France to solve the discrete event system problems [13]. The Min-Plus Algebra, after the initiation, has been used since and always gave the correct result [5-7]. But, the mathematical proof of mathematical correctness is still not yet well developed and written.

This paper present the result of mathematical proof of the correctness of Min-Plus Algebra for All Network's Shortest Paths Simultaneous Calculation.

## 2. Special Matrix Technique for Transportation Network Analysis

### 2.1 SMT for TNA

The SMT for TNA has ever been developed. First, it has been developed for Sparse Road Network Analysis. Later on, it is developed and tested to be used for other types of TNA. The technique consists of : network case statement, network model, matrix form convention and matrix calculation method. The Min-Plus Algebra is part of it [5-7].

### 2.2 Network Model

For the purpose of mathematical proof, a case of a road network was taken. The Network Model is therefore a simple link-node model [5-7]. A network model is presented in Figure 1 below.

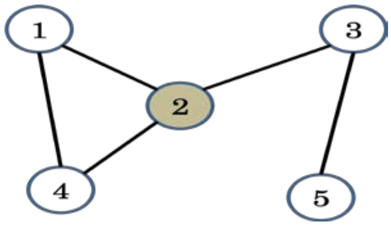


Figure 1 The Road Network Model

### 2.3 Matrix Representation of a Network

The matrix representation of a network is organized in : Basic Matrix, Expanded Matrix, Indicative Matrix. A matrix always a n×n special matrix. The diagonal cells are used to represent node data, while links data must be represented in non diagonal cells [5-7]. Two samples of matrix are presented in Table 1 as follow.

Table 1 The m.N and the m.L Matrices.

| m.N | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 |
| 2 | 0 | 3 | 0 | 0 | 0 |
| 3 | 0 | 0 | 1 | 0 | 0 |
| 4 | 0 | 0 | 0 | 1 | 0 |
| 5 | 0 | 0 | 0 | 0 | 1 |

| m.L | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 1 | 0 |
| 2 | 1 | 0 | 1 | 1 | 0 |
| 3 | 0 | 1 | 0 | 0 | 1 |
| 4 | 1 | 1 | 0 | 0 | 0 |
| 5 | 0 | 0 | 1 | 0 | 0 |

### 2.4 Min-Plus Algebra

The Min-Plus Algebra is a slight development of Max-Plus Algebra, in which the 'max operation' is changed into 'min operation'. The general operations and notations in Min-Plus Algebra are presented below [5,6].

$$a \oplus b = \min(a, b) \quad (1)$$

$$a \otimes b = a + b \quad (2)$$

$$|A| \otimes |B| = |C| \quad (3)$$

$$|C| = \bigoplus_{k=1-n} a_{ik} \otimes b_{kj} \quad (4)$$

$$c_{ij} = a_{i1} \otimes b_{1j} \oplus a_{i2} \otimes b_{2j} \oplus a_{i3} \otimes b_{3j} \oplus \ldots\ldots \oplus a_{in} \otimes b_{nj} \quad (5)$$

$$= \min((a_{i1} + b_{1j}), (a_{i2} + b_{2j}), (a_{i3} + b_{3j}), \ldots.., (a_{in} + b_{nj})) \quad (6)$$

$$|A|^{\otimes N} = |A| \underline{\otimes} |A|^{\otimes (N-1)} \quad (7)$$

### 2.5 Shortest Path Formula in SMT for TNA

Refering to the Special Matrix Technique convention, the Shortest Paths Simultaneous Calculation formulae is written below, by using the Min-Plus Algebra Principal [5,6].

$$m.SP = m.LL^{\otimes (N-1)} \quad (8)$$

The power of (N-1) is taken, because for a network with N nodes, for each pair of nodes, the maximum number of nodes to be passed from node o (origin) to node d (destination), without passing a certain node twice, is N. Therefore, the maximum number of links to be passed or the maximum number of steps can be made is (N-1) [5,6].

## 3. Correctness Proof

### 3.1 Method Development

Refering to Research Operation, shortest path calculation is in fact a matter of an integer case optimization. Several related method have been developed : total combination enumeration for shortest path calculation, general branch-and-bound technics for integer optimization and ordered combination with a reduced searching area [9,10,14,15]. These all deal with : champ of investigation (the whole possible combination down to feasible solution champ), combination process and fathoming rule [9,10,14,15].

The shortest path problem is an optimization case. Therefore, a mixture of : enumeration procedures and branch-and-bound technic type will be used. Hence, the Principal Algorithm Requirement to be fulfilled are : all itinerary combinations must be evaluated, each minimum result must be recorded, correct and appropriate fathoming technic, automatic presentation of the each absolute minimum as a result of calculation.

### 3.2 Correctness Proof

The proof must be done by evaluating wether yes or not, the Min-Plus Algebra for Shortest Path Simultanous Calculation satisfy the Algorithm Requirement formulated above.

**General Case.** To be easily understandable, a general network case, consists of 5 nodes and 8 links. The Network and its Link Length Matrix are presented in Figure 2 and Table 2 below.

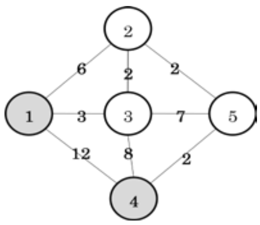


Figure 2 General Network Case

Table 2 Link Length Matrix

| m.LL | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0 | 6 | 3 | 12 | 999 |
| 2 | 6 | 0 | 2 | 999 | 2 |
| 3 | 3 | 2 | 0 | 8 | 7 |
| 4 | 12 | 999 | 8 | 0 | 2 |
| 5 | 999 | 2 | 7 | 2 | 0 |

**Shortest Path Calculation.** The most important, pertinent and interesting part of the Min-Plus Algebra Matrix Powering is an operation of Matrix Multiplication which incorporate the following formulae :

$$ll^{n+1}_{ij} = \min \{(ll^{n}_{i1}+ll^{n}_{1j}),(ll^{n}_{i2}+ll^{n}_{2j}),(ll^{n}_{i3}+ll^{n}_{3j}) \ldots (ll^{n}_{in}+ll^{n}_{nj})\} \quad (10)$$

where :

$ll^{n}_{ij}$ : shortest path length from node i to node j, for the $n^{th}$ powering step

The Min-Plus Algebra Calculation are presented in Table 3 below.

Table 3 The Min-Plus Algebra Calculation Table

| $m.LL^{\otimes 1}$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| **1** | 0 | 6 | 3 | 12 | 999 |
| **2** | 6 | 0 | 2 | 999 | 2 |
| **3** | 3 | 2 | 0 | 8 | 7 |
| **4** | 12 | 999 | 8 | 0 | 2 |
| **5** | 999 | 2 | 7 | 2 | 0 |

| $m.LL^{\otimes 1}$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| **1** | 0 | 6 | 3 | 12 | 999 |
| **2** | 6 | 0 | 2 | 999 | 2 |
| **3** | 3 | 2 | 0 | 8 | 7 |
| **4** | 12 | 999 | 8 | 0 | 2 |
| **5** | 999 | 2 | 7 | 2 | 0 |

| $m.LL^{\otimes 2}$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| **1** | 0 | 5 | 3 | 11 | 8 |
| **2** | 5 | 0 | 2 | 4 | 2 |
| **3** | 3 | 2 | 0 | 8 | 4 |
| **4** | 11 | 4 | 8 | 0 | 2 |
| **5** | 8 | 2 | 4 | 2 | 0 |

| $m.LL^{\otimes 3}$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| **1** | 0 | 5 | 3 | 10 | 7 |
| **2** | 5 | 0 | 2 | 4 | 2 |
| **3** | 3 | 2 | 0 | 6 | 4 |
| **4** | 10 | 4 | 6 | 0 | 2 |
| **5** | 7 | 2 | 4 | 2 | 0 |

| $m.LL^{\otimes 4}$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| **1** | 0 | 5 | 3 | 9 | 7 |
| **2** | 5 | 0 | 2 | 4 | 2 |
| **3** | 3 | 2 | 0 | 6 | 4 |
| **4** | 9 | 4 | 6 | 0 | 2 |
| **5** | 7 | 2 | 4 | 2 | 0 |

| $m.LL^{\otimes 5}$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| **1** | 0 | 5 | 3 | 9 | 7 |
| **2** | 5 | 0 | 2 | 4 | 2 |
| **3** | 3 | 2 | 0 | 6 | 4 |
| **4** | 9 | 4 | 6 | 0 | 2 |
| **5** | 7 | 2 | 4 | 2 | 0 |

**Calculation Detail and Connotation.** In order to be understandable easily, the calculation detail and connotation explication are presented for the calculation of shortest path length from node 1 to node 4 : $m.SP_{14}$. The discussion will be given for Step 0, 1 and 4 calculations. For each step the result are expressed by $m.LL_{14}^{\otimes 1}$ to $m.LL_{14}^{\otimes 4}$.

**Step 0.** Step 0 is the power 1 : $m.LL^{\otimes 1}$. Power 1 means make a one step itinerary for all pairs of nodes as origin and destination.

$$\begin{aligned} m.LL_{14}^{\otimes 1} &= m.LL_{14} \\ &= ll_{14} \\ &= 12 \end{aligned}$$

**Step 1.** Step 1 is the power 2 : $m.LL^{\otimes 2}$. Power 2 means make a two steps itinerary for all pairs of nodes as origin and destination. The calculation for Shortest Path Itinerary from Node 1 to Node 4 is presented below.

$$\begin{aligned} m.LL_{14}^{\otimes 2} &= m.LL_{14} \otimes m.LL_{14} \\ &= \min \{(ll_{11} + ll_{14}), (ll_{12} + ll_{24}), (\mathbf{ll_{13} + ll_{34}}), (ll_{14}+ll_{44}), (ll_{15} + ll_{54})\} \\ &= \min \{(0 + 12),(6+999),(\mathbf{3+8}),(12+0),(999+2)\} \\ &= 11 \end{aligned}$$

All itineraries are tested :

$ll_{11} - ll_{14}$ : real path : 1-1-4, path length = 0 + 12 = 12.

$ll_{12} - ll_{24}$ : real path : 1-2-4, path length = 6 + 999 = 999.

**$\mathbf{ll_{13} - ll_{34}}$ : real path : 1-3-4, path length = 3 + 8 = 11.**

$ll_{14} – ll_{44}$ : real path : 1-4-4, path length = 12 + 0 = 12.
$ll_{15} – ll_{54}$ : real path : 1-5-4, path length = 999 + 2 = 999.

**Step 3.** Step 3 is the power 4 : $m.LL^{\otimes 4}$. Power 4 means make a four steps itinerary for all pairs of nodes as origin and destination. The calculation for Shortest Path Itinerary from Node 1 to Node 4 is presented below.

$$m.LL_{14}^{\otimes 4} = m.LL^{\otimes 3}{}_{11} \underline{\otimes}\, m.LL_{14}$$
$$= \min \{(ll^3{}_{11} + ll_{14}), (ll^3{}_{12} + ll_{24}), (ll^3{}_{13} + ll_{34}), (ll^3{}_{14}+ll_{44}), (\mathbf{ll^3{}_{15} + ll_{54}})\}$$
$$= \min \{(0 +12),(5+999),(3+8),(10+0),(\mathbf{7+2})\}$$
$$= 9$$

Optimum itinerary :.
$ll_{15} – ll_{54}$ : $path^4$ : 1-5-4, path length = 7 + 2 = 9.
$path^2$ 12 : 1-3-2
$path^3$15 : 1-2-5
**real path : 1-3-2-5-4**

**Calculation Notes**. Principal Characteristics of Min-Plus Algebra Matrix Power Calculation are drawn and presented as follows :

- Each step of powering process, in terms of network, means adding one step forward in shortest path itinerary searching and is calculated in terms of path length.
- On each one step forward (each one step powering), the minimum path length is always calculated and taken, among the all existing combinations.
- Once the minimum absolute is gotten, the value remain the same throughout the next powering.
- Once, all cells are already filled with the minimum absolute value, the matrix cell values remain the same throughout the next powering, see the calculation : $m.LL^{\otimes 5} = m.LL^{\otimes 4}$.
- Achieving the whole absolute minimum can be reached in power (N-1) or before.

### 3.3 Algorithm Requirement Satisfaction

It has been shown that the whole Min-Plus Algebra power operation principal – matrix to matrix multiplication mechanism, for simultaneous shortest path length calculation, satisfy the whole requirement stated before.

- All Possible Itinerary Combinations are verified.
- Only Minimum Result is recorded for each cell for each step.
- The Fathoming Procedure is realized by always taken the minimum value of each step combination, thus also incorporating the fact that the optimum itinerary for a cell can be different across different steps.
- Once the Absolute Minimum Values is gotten, they always remain the same even though the powering process is continued..
- The Absolute Minimum are gotten automatically in the calculation end.

## 4. Conclusion

The research objective has been successfully achieved. Main conclusion points are presented as follows.

- The Min-Plus Algebra Matrix Powering can be used for shortest path length simultaneous calculation and mathematically correct.
- The Min-Plus Algebra Matrix Powering can be very easily done on spreadsheet type software and also be programmed by a certain programming language.

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