Analysis on Nonlinear Static Deflection and Natural Frequencies of a Thin Annular Plate with Initial Deformation

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Abstract. This paper presents analytical results on nonlinear static deflection and natural frequencies of a thin annular plate with initial deformation. Boundary conditions of the plate are inner-clamped and outer-free. At the inner boundary, the plate is assumed to be subjected to initial in-plane displacement. The coordinate function of the deflection in the radial direction is introduced as the power series while that in the circumferential direction is the sinusoidal function. The homogeneous and particular solutions of the stress function are obtained with the variation of constants method from the compatibility equation. Unknown constants in the homogeneous solution are determined in terms of deflection and parameters, with which the in-plane boundary condition is exactly satisfied. Applying the stress function into the governing equation, the equation of motion is obtained in terms of the deflection and the stress function. With the Galerkin procedure, the equation is reduced to a set of nonlinear ordinary differential equations. Based on the experimental results of deformed configuration under gravity and characteristics of restoring force, the initial deflection and initial in-plane displacement at the inner boundary are determined. As a result, the characteristics of restoring force are well identified. Furthermore, the natural frequencies of the plate show good agreement between the experiment and the analysis.

1. Introduction

Thin annular plates are used as fundamental components in industrial products such as sensors and actuators. The plates have initial deflection because of initial imperfection and effects of boundary conditions. Furthermore, initial deformation is generated by gravitational force. When the initial deformation is large, the plate easily shows nonlinear static deflection by an external load because lateral deflection is coupled with in-plane displacement. Furthermore, natural frequencies of the plate are different from those obtained by linear theory. Therefore, it is necessary to consider not only linear terms but also nonlinear terms in the analysis. To design the accurate industrial products, it is important to investigate the nonlinear static deflection and natural frequencies of the annular plate with initial deformation.

Analysis on the natural frequencies of thin annular plates under finite deformation considering nonlinear restoring force has been reported by Abdel-Rahman et. al. [1]. They analyzed natural frequencies of annular plates subjected to static loads taking axisymmetric modes into consideration.
Authors have shown experimental results on the static deflection and the natural frequencies of the annular plate with initial deformation [2,3]. However, it seems to authors that the analysis on the static deflection and the natural frequencies of an annular plate with initial deformation taking not only axisymmetric modes but also asymmetric modes into consideration has not been reported.

In this paper, the nonlinear static deflection and the natural frequencies of the annular plate with inner-clamped and outer-free boundary conditions are examined. The coordinate function in the radial direction is assumed to be power series while that in the circumferential direction is sine and cosine functions. As the compatibility equation is inhomogeneous equation, the stress function is expressed as sum of the homogeneous solution and the particular solution. Applying variables conversion to the compatibility equation, exact homogeneous solution is obtained. With the homogeneous solution, the particular solutions are obtained from the variation of constants method. The unknown coefficients in the homogeneous solution are determined from the in-plane boundary condition. Applying the Galerkin procedure to the governing equation of lateral deflection to which the stress function is substituted, the equation of motion in multiple-degree-of-freedom system are obtained. To validate the analysis, the static deflection and natural frequencies of the plate are compared to experimental ones.

2. Procedure of Analysis

The analytical model is shown in Fig. 1. The cylindrical coordinates are introduced to the annular plate. The \( r \)-axis and the \( \theta \)-axis are set to the radial and circumferential direction. The \( z \)-axis is perpendicular to the surface defined by \( r \)-axis and \( \theta \)-axis. The inner and outer radius of the plate is \( r_1 \) and \( r_2 \), respectively. The thickness of the plate is \( h \). The symbols \( \rho, E \) and \( \nu \) represent the mass density, the Young’s modulus and the Poisson’s ratio, respectively.

\[
\xi = \frac{r}{r_2}, [u, v] = \frac{r_2}{h^2} [U, V], [w, w_0] = \frac{1}{h} [W, W_0], f = \frac{F}{D},
\]
\[
[n_r, n_\theta, n_{r\theta}] = \frac{r_2}{D} [N_r, N_\theta, N_{r\theta}], p_s = \frac{\rho r_2^4}{D}, q_s = \frac{r_2^2}{Dh} Q_s, \tau = \Omega_0 t
\] (1)

The symbol \( \xi \) denotes the non-dimensional coordinate in the radial direction. The symbols \( u \) and \( v \) are in-plane displacement in radial and circumferential direction, respectively. The symbols \( w \) and \( w_0 \) denote non-dimensional deflection and initial deflection, respectively. The symbol \( f \) denotes non-dimensional stress function. The symbols \( n_r, n_\theta \) and \( n_{r\theta} \) are the stress resultants. These stress resultants
can be expressed in terms of stress function. The symbol \( p_s \) is intensity of distributed force generated by gravitational acceleration \( g \). The symbol \( q_s \) denotes non-dimensional concentrated force. The symbol \( \tau \) denotes non-dimensional time. For practical reason, variable conversion is adapted to the coordinate \( \xi \). A new variable \( \eta \) is defined as \( \eta = \ln \xi \). Hence, the inner and outer radius of the plate correspond to \( \eta_1 = \ln \xi_1 \) and \( \eta_2 = \ln \xi_2 \), respectively.

2.2 Governing Equations and Boundary Conditions

With the Hamilton’s principle, the governing equation is obtained. The stress function \( f \) is introduced to the governing equation in which the inertia in the in-plane direction is neglected. From the relation between the strain and the stress function, the compatibility equation is obtained. The governing equation and the compatibility equation are shown as follows.

\[
\begin{align*}
-\int_{\tau_0}^{\tau_1} \left[ \int_{\eta_1}^{\eta_2} 2\pi G(f, w) \delta w e^{2\eta} d\eta d\theta + B_1 + B_2 \right] d\tau &= 0 \quad (2) \\
\nabla^4 f &= ce^{-4\eta} \left[ (w_{,\eta\theta} - w_{,\theta})^2 - (w_{,0\eta\theta} - w_{,0\theta})^2 \right] \\
&\quad - (w_{,\eta\eta} - w_{,\eta})(w_{,\eta} + w_{,\theta\theta}) + (w_{,\eta\eta} - w_{,0\eta})(w_{,\eta} + w_{,0\theta}) \\
&= 0 \quad (3)
\end{align*}
\]

In Eq. (2), \( G(f, w) \) corresponds to the equation of motion. The terms \( B_1 \) and \( B_2 \) correspond to boundary conditions of lateral and in-plane direction, respectively. In Eq. (3), the symbol \( c \) is defined as \( c = 12(1 - v^2) \). The terms of \( G(f, w) \), \( B_1 \) and \( B_2 \) are shown below.

\[
\begin{align*}
G(f, w) &= w_{,tt} - e^{-4\eta} \left\{ (f_{,\eta\eta} - 2f_{,\eta} - f_{,\theta\theta})w_{,\eta} + (f_{,\eta} + f_{,\theta\theta})w_{,\eta\eta} + (f_{,\eta\eta} - f_{,\eta})w_{,\theta\theta} \\
&\quad + 2(f_{,\eta\theta} - f_{,\theta})w_{,\theta\theta} \right\} + \nabla^4 (w - w_0) \\
&\quad - p_s - q_s \delta(\eta - \eta_\text{qs}) \delta(\theta - \theta_\text{qs}) = 0 \quad (4)
\end{align*}
\]

\[
\begin{align*}
B_1 &= \left[ \int_{\eta_1}^{\eta_2} \left\{ (e^{-\eta} n_r w_{,\eta} + n_{\eta\theta} w_{,\theta} + v_r) \delta w - e^{-\eta} n_m \delta w_{,\eta} \right\} e^\eta d\theta \right]_{\eta = \eta_1}^{\eta = \eta_2} \quad (5)
\end{align*}
\]

\[
\begin{align*}
B_2 &= \left[ \int_{\eta_1}^{\eta_2} (n_r \delta u + n_{\theta\theta} \delta v) e^\eta d\theta \right]_{\eta = \eta_1}^{\eta = \eta_4} \quad (6)
\end{align*}
\]

In Eq. (5), the symbols \( m_r \) and \( v_r \) denote the non-dimensional bending moment and equivalent shear force.

To derive the nonlinear equation of motion of the plate, the deflection and the initial deflection is assumed as below.

\[
[w, w_0] = \sum_j \sum_N \xi_{jn} \left[ \frac{[\xi_{jn}, \xi_{jn}]}{\cos N\theta + [\xi_{jn}\delta N_0, \xi_{jn}\delta N_0]} \sin N\theta \right]_{j, N = 0, 1, 2, ...} \quad (7)
\]

In the above equation, the symbol \( \xi_{jn} \) and \( \xi_{jn} \) are unknown time functions and the symbol \( \xi_{jn} \) and \( \xi_{jn} \) are given constants to express the initial deflection. The term of \( \xi_{jn} \) is coordinate function in the radial direction and is expressed as follows.

\[
\xi_{jn} = \sum_k c_{jkN} e^{(k + \delta_{k4})\eta}, \quad (k = 0, 1, 2, 3, 4) \quad (8)
\]

In Eq. (8), \( c_{jkN} \) is coefficients which are chosen to satisfy the boundary condition of lateral deflection and \( e^{(k + \delta_{k4})\eta} \) corresponds to the power series in the coordinates \( \xi \).

As the compatibility equation is non-homogeneous equation, the stress function is expressed as the sum of the homogeneous solution and the particular solution. From Eq. (3), the homogeneous solution is obtained exactly [4]. Substituting Eq. (7) into Eq. (3), the particular solution of the stress function is obtained by using the variation of constants method. The homogeneous solution contains unknown coefficients which are exactly determined by in-plane boundary conditions.
In this paper, the plate is assumed to be subjected to initial in-plane displacement radially and circumferentially at the inner edge. The in-plane displacement \( u \) and \( v \) are derived from relation among the deflection, the in-plane displacement and the stress function. Taking the initial in-plane displacement into consideration, boundary condition is expressed as follows.

\[
\eta = \eta_1: u = u_t, v = v_t \\
\eta = \eta_2: n_r = n_r\theta = 0
\]

(9)

The initial in-plane displacement \( u_t \) and \( v_t \) are introduced as the known function. The function \( u_t \) and \( v_t \) are assumed as the Fourier series as shown below.

\[
u_t = u_0^{[c]} + \sum_{M=1}^{c} \left( u_M^{[c]} \cos M\theta + u_M^{[s]} \sin M\theta \right) \\
v_t = -\sum_{M=1}^{s} \left( v_M^{[s]} \sin M\theta - v_M^{[s]} \cos M\theta \right)
\]

(10) (11)

The symbols \( u_M^{[c]} \) and \( u_M^{[s]} \) are the given constant values and the \( M \) is the wavenumber in the circumferential direction.

With the deflection \( w \) and the stress function \( f \), the in-plane displacement and the stress resultants at the boundaries are expressed to determine the unknown coefficient.

2.3 Reduction to multiple-degree-of-freedom system

The deflection and the stress function are substituted into Eq. (2). Applying the Galerkin procedure to the governing equations, the equation is reduced to multi-degree-of-freedom system. The set of nonlinear ordinary differential equations in terms of \( \ddot{c}_{pN} \) and \( \ddot{s}_{pN} \) is obtained. To simplify the equations, the unknown time functions \( \ddot{c}_{pN} \) and \( \ddot{s}_{pN} \) are combined to \( \ddot{b}_j \) and the given coefficients for initial deflection \( \ddot{c}_{pN} \) and \( \ddot{s}_{pN} \) are combined to \( \ddot{b}_j \). The set of the nonlinear ordinary differential equations in terms of \( \ddot{b}_j \) is shown as below.

\[
\sum_j B_{ij} \ddot{b}_{j,rt} + \sum_j \dot{C}_{ij} \dot{b}_j + \sum_j \sum_k \sum_l E_{ijkl} \ddot{b}_j \ddot{b}_k \ddot{b}_l - \ddot{F}_i - \ddot{G}_i p_s - \ddot{H}_i q_s = 0 \\
(i, j, k, l = 1, 2, ...)
\]

(12)

In the foregoing, the symbol \( \ddot{B}_{ij} \) is the coefficient of inertia term. The symbols \( \ddot{C}_{ij} \) and \( \ddot{E}_{ijkl} \) represent the coefficients corresponding to the restoring force of linear and cubical terms, respectively. The symbol \( \ddot{F}_i \) is the coefficient calculated from the initial deflection \( \ddot{b}_j \). The symbol \( \ddot{G}_i \) and \( \ddot{H}_i \) are the coefficients related gravitational force and concentrated force.

The deflection is assumed to be the sum of static deflection and dynamic deflection. The dynamic deflection is measured from static equilibrium position. Symbols \( \ddot{b}_j \) and \( \ddot{b}_j \) are introduced as the unknown constant of the static deflection and the unknown time function of the dynamic deflection, respectively. The set of the equation in terms of \( \ddot{b}_j \) and \( \ddot{b}_j \) is obtained from Eq. (12) as below.

\[
\sum_j C_{ij} \ddot{b}_j + \sum_j \sum_k \sum_l E_{ijkl} \ddot{b}_j \ddot{b}_k \ddot{b}_l - \ddot{F}_i - \ddot{G}_i p_s - \ddot{H}_i q_s = 0 \\
\sum_j \dddot{B}_{ij} \dddot{b}_j + \sum_j \dddot{C}_{ij} \dddot{b}_j + \sum_j \sum_k \dddot{E}_{ijkl} \dddot{b}_j \dddot{b}_k + \sum_j \sum_k \sum_l \dddot{E}_{ijkl} \dddot{b}_j \dddot{b}_k \dddot{b}_l = 0
\]

(13) (14)

In Eq. (14), the symbols \( \dddot{C}_{ij} \) and \( \dddot{E}_{ijkl} \) represent the coefficients corresponding to the restoring force of linear and quadratic terms, respectively, which depend on the static deflection \( \ddot{b}_j \). With the first and second terms in Eq. (14), the linear natural frequencies of the plate under nonlinear finite deformation are obtained.
2.4 Identification of initial in-plane displacement at the inner edge

In the experiments, it is difficult to measure accurately the initial deflection and the initial in-plane displacement at the clamped boundary. Therefore, the initial deflection and the in-plane displacement are identified by comparing analytical and experimental results of static deformation under gravity and characteristics of restoring force by a concentrated load. In the identification, the initial deflection $\tilde{b}_j$, and the initial in-plane displacement $u_i, v_i$ are the unknowns in Eq. (13). These unknowns are identified as the following procedure. First, the initial values of in-plane displacement $u_i$ and $v_i$ are chosen. Approximating the configuration of the initial deformation in the experiment with linear combinations of the coordinate function, the symbol $\tilde{b}_j$ can be obtained. Then, substituting the assumed $u_i, v_i$ and $\tilde{b}_j$ into Eq. (13), the symbol $\tilde{b}_j$ is calculated by solving coupled cubic equations. Next, to determine the initial in-plane displacement, the characteristics of restoring force of the plate under a concentrated load measured in the experiment are used. The in-plane displacements are identified by numerically minimizing the residual between static deflections at selected several values of concentrated force on the experiment and analysis based on the assumed values of in-plane displacement.

In this paper, the wavenumber $M$ in the initial in-plane displacement is assumed to be $M = 0, 2, 4$ because it is found that these wavenumbers strongly affect the change in natural frequencies in our preliminary analysis.

3. Results and Discussion

To validate the analysis, the analytical results are compared with the experiments [3] by the authors. The restoring force of the plate is shown in Fig. 2. The concentrated force is loaded at the position $(\zeta, \theta) = (0.92, 0)$ and the static deflection is measured at the two positions $(\zeta, \theta) = (0.52, 0)$ and $(0.60, \pi)$, which are denoted by circles and triangles, respectively. To identify the initial in-plane displacement comparing analytical and experimental results, the values of concentrated load in the experiment are selected at eight points, which are also denoted in Fig. 2 with numbers 1 to 8. The analytical results are also shown in the figure with solid and dashed lines. The analytical results show fairly good agreement to the experimental ones.

![Restoring force of the plate](image)

Fig. 2. Restoring force of the plate

The linear natural frequencies and the corresponding vibration modes are shown in Table 1. In Table 1, the symbol $i$ indicates the number of nodal diameters. To distinguish the modes with same number of nodal diameters, symbols “a” and “b” are added to the number. Dashed lines in the modal patterns
indicate the nodal diameters. The linear natural frequencies obtained by the experiment and the analysis are relatively close. In the analysis, direction of nodal lines in mode 1a and 1b are different from the experimental ones. However, the modal patterns of higher modes are relatively similar to each other.

![Image](image.png)

### Table 1. Linear natural frequencies and corresponding vibration modes

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<th>(a)Experiment</th>
<th></th>
<th>(b)Analysis</th>
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<td>$\omega_i$</td>
<td>Modal pattern</td>
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<td>0</td>
<td>1b</td>
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</table>

### 4. Conclusion

The analysis on nonlinear static deflection and natural frequencies of the thin annular plate with initial deformation is reported. In the analysis, the initial deflection and the initial in-plane displacement are taken into account. The initial deflection and the initial in-plane displacement are identified by comparing analytical and experimental results of static deformation under gravity and characteristics of restoring force by a concentrated force. As a result, characteristics of the restoring force are well identified. The natural frequencies and the model patterns of higher modes in the analysis show relatively good agreement with the experiment.

### References


