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Division by Zero Calculus and Singular Integrals

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Abstract. In this paper, we will introduce the formulas $\log 0 = \log \infty = 0$ (not as limiting values) in the meaning of the one point compactification of Aleksandrov and their fundamental applications, and we will also examine the relationship between Laurent expansion and the division by zero. Based on those examinations we give the interpretation for the Hadamard finite part of singular integrals by means of the division by zero calculus. In particular, we will know that the division by zero is our elementary and fundamental mathematics.

1. Introduction

By a natural extension of the fractions

$$\frac{b}{a} \tag{1}$$

for any complex numbers a and b, we found the simple and beautiful result, for any complex number b

$$\frac{b}{0} = 0, \tag{2}$$

incidentally in [1] by the Tikhonov regularization for the Hadamard product inversions for matrices and we discussed their properties and gave several physical interpretations on the general fractions in [2] for the case of real numbers. The result is very special case for general fractional functions in [3].

The division by zero has a long and mysterious story over the world (see, for example, H. G. Romig [4] and Google site with the division by zero) with its physical viewpoints since the document of zero in India on AD 628. However, Sin-Ei Takahasi ([2]) established a simple and decisive interpretation Eq. (2) by analyzing the extensions of fractions and by showing the complete characterization for the property Eq. (2).

In this paper, we will show that $\log 0 = \log \infty = 0$, by the division by zero $z/0 = 0$ and their fundamental applications. We will give the interpretation for the Hadamard finite part of singular integrals by means of the division by zero calculus.

2. Calculation by division by zero

As the number system containing the division by zero, the Yamada structure is complete in [5]. However, for applications of the division by zero to functions, we will need the concept of division by zero calculus for the sake of uniquely determination of the results.

For example, for the typical linear mapping

$$W = \frac{z-i}{z+i} \quad , \tag{3}$$

it gives a conformal mapping on $\{\mathbf{C} \setminus \{-i\}\}$ onto $\{\mathbf{C} \setminus \{1\}\}$ in one to one and from

$$W = 1 + \frac{-2i}{z-(-i)} \quad , \tag{4}$$

we see that $-i$ corresponds to 1 and so the function maps the whole $\{\mathbf{C}\}$ onto $\{\mathbf{C}\}$ in one to one.

Meanwhile, note that for

$$W = (z-i) \cdot \frac{1}{z+i} \quad , \tag{5}$$

we should not enter $z = -i$ in the way

$$[(z-i)]_{z=-i} \cdot [\frac{1}{z+i}]_{z=-i} = -2i \cdot 0 = 0. \tag{6}$$

Therefore, we will need to introduce the division by zero calculus: For any formal Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z-a)^n. \tag{7}$$

We obtain the identity, by the division by zero

$$f(a) = C_0. \tag{8}$$

Note that here, there is no problem on any convergence of the expansion Eq. (7) at the point $z = a$.

For the correspondence Eq. (8) for the function $f(z)$, we will call it **the division by zero calculus**.

We will give typical examples. For the typical function $(\sin x)/x$, we have

$$\frac{\sin x}{x}(0) = \frac{\sin 0}{0} = \frac{0}{0} = 0,$$

however, by the division by zero calculus, we have, for the function $(\sin x)/x$

$$\frac{\sin x}{x}(0) = 1,$$

that is more reasonable in analysis.

For the elementary function

$$f(x) = \frac{x}{x-1},$$

We should not consider that for the point $x = 1$

$$f(1) = \frac{1}{0} = 0,$$

but we should consider it as $f(1) = 1$ by the division by zero calculus.

For some relation of smoothness of functions and the division by zero calculus, the following idea will be important for some interpolation of the Hadamard finite part of singular integrals and the division by zero calculus:

For a smooth function $f(x)$ of class C^n for $n \geq 1$, from the Taylor expansion around $x = a$

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n)}(c)}{n!}(x-a)^n, \tag{9}$$

$$a < c < x \quad or \quad a > c > x,$$

We obtain, by the division by zero

$$\left[\frac{f(x)}{(x-a)^m}\right]_{x=a} = \begin{cases} \dfrac{f^{(n)}(c)}{n!} & (m=n) \\ 0 & (m>n) \\ \dfrac{f^{(m)}(a)}{m!} & (m<n). \end{cases} \tag{10}$$

This formula shows the relation of derivatives and division by zero calculus; the limiting values $\lim_{x\to a}$ and the values at the point of the formal singular point $x = a$.

3. Introduction of formulas $\log 0 = \log \infty = 0$

For any fixed complex number a, we will consider the sector domain $\Delta_a(\alpha, \beta)$ defined by

$$0 \le \alpha < \arg(z-a) < \beta < 2\pi$$

on the complex z plane and we consider the conformal mapping of $\Delta_a(\alpha, \beta)$ into the complex W plane by the mapping,

$$W = \log(z-a). \tag{11}$$

Then, the image domain is represented by

$$S(\alpha, \beta) = \{W; \alpha < \Im W < \beta\}.$$

Here, we will note the space structure by the division by zero.

We will be able to see the whole Euclidean plane by the stereographic projection into the Riemann sphere --- *In the Euclidean plane, there does not exist the point at infinity*.

*The infinity is not a number, but it is an ideal space point of **the one point compactification by Aleksandrov**.*

The behavior of the space around the point at infinity may be considered (may be defined) by that around the linear transform $W = 1/z$ ([6]). We thus see that

$$\lim_{z\to\infty} z = \infty, \tag{12}$$

however,

$$[z]_{z=\infty} = 0, \tag{13}$$

by the division by zero. Here, $[z]_{z=\infty}$ denotes the value of the function $W = z$ at the topological point at the infinity in *one point compactification by Aleksandrov*. The difference of Eq. (12) and Eq. (13) is very important as we see clearly by the function $W = 1/z$ and the behavior at the origin. The limiting value to the origin and the value at the origin are different. For surprising results, we will state the property in the real space as follows:

$$\lim_{x\to+\infty} x = +\infty, \quad \lim_{x\to-\infty} x = -\infty, \tag{14}$$

however,

$$[x]_{x=+\infty} = 0, \quad [x]_{x=-\infty} = 0. \tag{15}$$

Of course, two points $+\infty$ and $-\infty$ are the same point as the point at infinity. However, $\pm$ will be convenient in order to show the approach directions. We gave also many evidences by analytic geometry in the Euclidean space for these properties ([7]).

Next, two lines $\{W; \Im W = \alpha\}$ and $\{W; \Im W = \beta\}$ usually were considered as having the common point at infinity, however, in the division by zero, the point is represented by zero.

Therefore, $\log 0$ and $\log \infty = 0$ **should be defined as zero**. Here, $\log \infty$ is precisely given in the sense of $[\log z]_{z=\infty}$. However, the properties of the logarithmic function should not be expected more, we should consider the value only. For example,

$$\log 0 = \log(2 \cdot 0) = \log 2 + \log 0$$

is not valid. We can apply the result $\log 0 = 0$ for many cases as in these papers [8, 9].

4. Finite parts of Hadamard in singular integrals

Singular integral equations are presently encountered in a wide range of mathematical models, for instance in acoustics, fluid dynamics, elasticity and fracture mechanics. Together with these models, a variety of methods and applications for these integral equations has been developed. See, for example, [10, 11, 12, 13].

For the numerical calculation of this finite part, see [14], and there, they gave an effective numerical formulas by using the DE formula. See also its references for various methods.

Let $F(x)$ be an integrable function on an interval (c,d). The function $F(x)/(x-a)^n$ $(n=1,2,3\ldots, c<a<d)$ are, in general, not integrable on (c,d). However, for any $\epsilon > 0$, of course, the functions

$$\left(\int_c^{a-\epsilon} + \int_{a+\epsilon}^d\right)\frac{F(x)}{(x-a)^n}dx \tag{16}$$

are integrable. For an integrable function $\varphi(x)$ on (a,d), we assume the Tayler expansion

$$F(x) = \sum_{k=0}^{n-1}\frac{F^{(k)}(a)}{k!}(x-a)^k + \varphi(x)(x-a)^n. \tag{17}$$

Then, we have

$$\int_{a+\epsilon}^d \frac{F(x)}{(x-a)^n}dx = \sum_{k=0}^{n-2}\frac{F^{(k)}(a)}{k!(n-k-1)}\frac{1}{\epsilon^{n-k-1}} - \frac{F^{(n-1)}(a)}{(n-1)!}\log\epsilon$$
$$+\left\{-\sum_{k=0}^{n-2}\frac{F^{(k)}(a)}{k!(n-k-1)}\frac{1}{(d-a)^{n-k-1}} + \frac{F^{(n-1)}(a)}{(n-1)!}\log(d-a) + \int_{a+\epsilon}^d \varphi(x)dx\right\}.$$

Then, the last term $\{....\}$ is the finite part of Hadamard of the integral

$$\int_a^d \frac{F(x)}{(x-a)^n}dx \tag{18}$$

and is written by

$$\text{f. p.}\int_a^d \frac{F(x)}{(x-a)^n}dx;$$

that is, precisely

$$\text{f. p.}\int_a^d \frac{F(x)}{(x-a)^n}dx := \lim_{\epsilon\to+0}\left\{\int_{a+\epsilon}^d \frac{F(x)}{(x-a)^n}dx - \sum_{k=0}^{n-2}\frac{F^{(k)}(a)}{k!(n-k-1)}\frac{1}{\epsilon^{n-k-1}} + \frac{F^{(n-1)}(a)}{(n-1)!}\log\epsilon\right\}. \tag{19}$$

We do not take the limiting $\epsilon \to +0$, but we put $\epsilon = 0$, in Eq. (19), then we obtain:

$$\text{f. p.}\int_a^d \frac{F(x)}{(x-a)^n}dx = \int_a^d \frac{F(x)}{(x-a)^n}dx. \tag{20}$$

Division by zero will give the natural meaning (**definition**) for the above two integrals. Of course,

$$\text{f. p.}\int_c^d \frac{F(x)}{(x-a)^n}dx := \text{f. p.}\int_c^a \frac{F(x)}{(x-a)^n}dx + \text{f. p.}\int_a^d \frac{F(x)}{(x-a)^n}dx. \tag{21}$$

When $n=1$, the integral is the Cauchy principal value.

In particular, for the expression Eq. (19), we have, missing $\log\epsilon$ term, for $n \ge 2$

$$\text{f. p.}\int_c^d \frac{F(x)}{(x-a)^n}dx = \lim_{\epsilon\to+0}\left\{\left(\int_c^{a-\epsilon} + \int_{a+\epsilon}^d\right)\frac{F(x)}{(x-a)^n}dx - \sum_{k=0}^{n-2}\frac{F^{(k)}(a)}{k!(n-k-1)}\frac{1+(-1)^{n-k}}{\epsilon^{n-k-1}}\right\}. \tag{22}$$

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