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A Finite Difference Method for a Thin Film Model

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Abstract. We consider a fluid flowing on an inclined channel. The fluid depth is much smaller than the unity length, so that it can be formulated based on lubrication theory into a single equation of the fluid depth. Since the model is partial differential equation and strongly non-linear, a numerical approach is proposed to get the solution. From the linearized model, the solution presents surface wave propagating with decreasing the amplitude and a forward time central space is conditionally stable. Therefore, the numerical solution is studied to simulate the wave propagation. The character of the wave is confirmed.

1. Introduction

Thin-film fluid is considered on an inclined channel, so that the gravity drives the fluid going down. Since the thickness of the fluid is much smaller than the unity of the length, the lubrication theory can be applied as the momentum equation. Together with the continuity equation and some boundary conditions, the governing equation can be expressed as a single equation of the thickness of the fluid. Wiryanto and Febrianti [1] derived the equation. Similar model can also be seen in King, at. al. [2] for steady flow but involving air flow above the thin film in different direction. They obtained the model in an integro-different equation as the interaction between the fluid and the air. Analytical work shows that the fluid surface presents periodic wave, and it is confirmed by its numerical solution.

In contrast with steady model, we are interested in model of unsteady but without upper air flow. The equation is strongly non-linear of the variable of the thickness of the fluid depth presenting wave propagation. Wiryanto [3] then analyzed the character of the wave through the linearized model. He obtained that the wave in its propagation is damped. In his analysis, the solution is presented as monochromatic wave. However, the amplitude depends on time with exponent form of negative coefficient, so that it decreases by increasing time.

From that wave character of the linearized model, we are interested in this study to observe the solution of the original equation, non-liner form. Numerical approach is used to get the solution. Forward-time central space is chosen as the method, since the method is stable with a certain condition. King, et. al. [2] indicated that the solution tends to a type of wave called roll wave. This character can be also obtained by Fauzan and Wiryanto [4] for shallow water model. The periodic wave propagates with deforming to roll wave for a certain condition, relating to the average depth and velocity. Study of that type of wave can be referred some in [5, 6, 7]. In our work, the surface wave of thin-film flow indicates deforming to roll wave by tending the profile sharping at the front wave.

2. Formulation

The sketch of the flow and coordinates are shown in Fig. 1. The fluid flows on an inclined channel of angle θ . The thickness is h mesured from the bottom of the channel. We choose the horizontal x –axis is along the bottom of the channel and the vertical y –axis is perpendicular to x –axis, so that the gravity is proyected to those axis effected to the movement of the fluid particle.



Fig. 1. Sketch of the flow and coordinates

Following Wiryanto and Febrianti [1], when the fluid thickness is disturbed by h(x, 0) it changes satisfying the model

$$h_t + \frac{\rho}{3\mu} \Big[-h^3 h_x g \cos\theta + h^3 g \sin\theta \Big]_x = 0 \tag{1}$$

where ρ is the fluid density, μ is viscosity, g is the accelation of gravity. The equation was derived based on the continuity equation and Navier Stokes equation as the momentum equation. Since the thickness is much smaller than the horizontal unity, lubrication theory can simplify to momentum equation. When they are solved by involving the boundary conditions, un-slip condition along the bottom. kinematics condition and zero pressure at the surface, we come to (1).

Near constant solution h_0 , we suppose the solution in $h(x,t) = h_0 + \varepsilon \eta(x,t)$ for small ε . When it is substituted in (1) and we take the first order we have a linear equation. Wiryanto [3] then analyzed that the equation has solution with deceasing the amplitude by increasing time. Meanwhile, a finite difference method of forward-time central space (FTCS) for the linear equation is stable when $\Delta t < \frac{\Delta x^2}{2b}$. Δt , Δx are the time step and the length of the space discretization. $b = \rho g \cos \theta / 3\mu$ is

constant.

Based on that analysis, we then develop the numerical procedure for (1) by FTCS considering the stability condition. We discretize the space x into J subintervals with end points $x_j = j \Delta x$, for j = 0, 1, 2, ..., J, so that we define $h_j^n \approx h(x_j, t_n)$. After discretization, the finite difference equation for (1) can be expressed in explicit form

$$h_{j}^{n+1} = h_{j}^{n} - \frac{\Delta t}{\Delta x} \left[\left(h_{j-1}^{n} \right)^{3} \left(a \frac{h_{j}^{n} - h_{j-2}^{n}}{2 \Delta x} - b \right) - \left(h_{j+1}^{n} \right)^{3} \left(a \frac{h_{j+2}^{n} - h_{j}^{n}}{2 \Delta x} - b \right) \right]$$
(2)

We denote $a = \rho g \sin \theta / (3\mu)$. For calculating h_j^{n+1} we need 5 values of h at the previous step time. When j = 0 and j = J we need two valued of h outside of the observation domain. We provide by considering the periodic solution. So we use condition $h_{-1}^{n+1} = h_j^n$, $h_{-2}^{n+1} = h_{J-1}^n$ for the left boundaries and $h_{J+1}^{n+1} = h_0^{n+1}$, $h_{J+2}^{n+1} = h_1^{n+1}$ for the right boundaries. These conditions express that we follow the wave, and we expect that we can see the wave deformation.

3. Numerical Solution

In this section, we present the numerical solution of the model (1). Most of our calculations uses g = 10, $\rho = 1$, and various values μ and angle θ , but we choose small θ . For $\theta = 5^{\circ} \approx 0.017$ radian

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and $\mu = 2$ following the stability criteria, for $\Delta x = 0.1$ the step time is $\Delta t < 0.016$. In our calculation we use $\Delta t = 0.01$.



Fig. 2. Plot of some free surfaces h_j^n , each curve presenting surface for different time *n*, by shifting upward to show the simulation and the deformation of the surface from initial condition $h_i^0 = 0.2 + 0.05 \sin(0.25 \pi x_i)$



Fig. 3. Similar to Figure 2, h_j^n calculated using the initial condition $h_j^0 = 0.2 + 0.1 \operatorname{sech}[0.15(x_j - 20)]$

As the initial value the surface is in sinusoidal $h_j^0 = 0.2 + 0.05 \sin(0.25 \pi x_j)$ at the observation domain 0 < x < 100. For the linear model, this initial value gives similar profile with smaller amplitude. The effect of nonlinearity is shown in Fig. 2. We plot some free surfaces in the same plane for different time by shifting upward for higher time. We can see the deformation of the surface profile, from sinusoidal to sharper at the front wave, followed by decreasing the amplitude. In Fig. 3, we show another solution of (1) with initial value $h_j^0 = 0.2 + 0.1 \operatorname{sech}[0.15(x_j - 20)]$. The wave propagates with changing the form.

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4. Conclusion

A finite difference method FTCS has been applied to the model of thin film flow. The stability condition for the linear model is valid for non-linear one, so that the method can be used to simulate the wave propagation. The effect of the non-linearity has been shown by changing the form.

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